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LETTER TO THE EDITOR

# On the relationship between long-time correlations and replica correlations in disordered systems

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**Abstract.** A proof is given of the equivalence between the replica correlations, ($\langle \sigma_i^\alpha \sigma_i^\beta \rangle = q$, which have been used by Edwards and Anderson to construct a theory of spin-glass equilibrium properties) and the correlations at long-time predicted by the dynamical equations.

The authors have frequently heard the comment that the order parameter for a spin-glass, introduced by Edwards and Anderson, was not obviously the same as the long-time correlation function. We thought it worthwhile, therefore, to publish a proof of this equivalence.

Consider a system described by a Hamiltonian $H$, which depends on fast and slow modes of motion in a dynamical system (see Edwards (1976)). If $x$ denotes the set of fast modes and $X$ the set of slow modes then:

$$H = H(x, X). \tag{1}$$

In the analysis of a spin-glass given by Edwards and Anderson (1975) the set $X$ are actually fixed and random with a given probability distribution. The equation of motion for the phase distribution function, $f$, which describes the system dynamics is:

$$\partial f/\partial t + [H, f] = 0. \tag{2}$$

Similarly an equation of motion for the propagator $G$ corresponding to equation (2) may be written:

$$\partial G/\partial t + [H, G] = \delta. \tag{3}$$

As is known, $G$ can be expressed in terms of the eigenfunctions and eigenvalues of the operator defining equation (2). Thus:

$$G = \sum_n \int \chi_n(x, X)\Psi_n(x', X) \exp[-E_n(t-t')], \qquad t > t' \tag{4}$$

where, in particular:

$$\Psi_0 = 1 \qquad \text{and} \quad \chi_0 = \exp[(F-H)/kT] \tag{5}$$

in which $F$ is the free energy of the system and $\chi_0$ is the Gibbs equilibrium solution to the dynamical equation (2). Next we define a correlation function $C$ by:

$$
\begin{aligned}
C(t) &= \langle\langle x(t)x(0)\rangle\rangle \\
&= \int \mathrm{d}X\, P(x) \int \mathrm{d}x' x' \chi_0(x') \int \mathrm{d}x\, x G(x, x'; t) \\
&= \left\langle \int \mathrm{d}x' x' \chi_0(x') \int \mathrm{d}x\, x G(x, x'; t) \right\rangle
\end{aligned} \tag{6}
$$

in which $P(x)$ is the probability of finding $X$. Taking the limit $t \to \infty$:

$$
\lim_{t\to\infty} C(t) = C(\infty) = \left\langle \int \mathrm{d}x\, x\chi_0(x) \int \mathrm{d}x'\, x'\chi_0(x') \right\rangle \tag{7}
$$

which can be written:

$$
C(\infty) = \left\langle \left\{ \int \mathrm{d}x\, x \exp[-H(x, X)/kT] \right\}^2 \Big/ \left\{ \int \mathrm{d}y \exp[-H(y, X)/kT] \right\}^2 \right\rangle \tag{8}
$$

since $\exp(-F/kT) = \int \mathrm{d}\Omega \exp[-H/kT]$, where $\int \mathrm{d}\Omega$ signifies an integration over accessible phase space. The next step in the proof is to consider replicas defining an order parameter $q$:

$$
\begin{aligned}
q &= \text{`}\langle x_1 x_2\rangle\text{'} \\
&= \lim_{N\to 0} \left\langle \int \mathrm{d}x_1\, x_1 \exp[-H(x_1, X)/kT] \int \mathrm{d}x_2\, x_2 \exp[-H(x_2, X)/kT] \right. \\
&\quad \left. \times \left\{ \int \ldots \int \prod_{j=3}^{N} \mathrm{d}x_j \exp\left[ -\sum_{\alpha=3}^{N} H(x_\alpha, X)/kT \right] \right\} \right\rangle .
\end{aligned} \tag{9}
$$

Clearly

$$
\begin{aligned}
q &= \lim_{N\to 0} \left\langle \left\{ \int \mathrm{d}x\, x \exp[-H(x, X)/kT] \right\}^2 \left\{ \int \ldots \int \prod_{j=3}^{N} \mathrm{d}x_\alpha \exp\left[ -\sum_{\sigma=3}^{N} H(x_\alpha, X)/kT \right] \right\} \right\rangle \\
&= \lim_{N\to 0} \left\langle \frac{\{\int \mathrm{d}x\, x \exp[-H(x, X)/kT]\}^2}{\{\int \mathrm{d}y \exp[-H(y, X)/kT]\}^2} \left\{ \int \mathrm{d}z \exp[-H(z, X)/kT] \right\}^N \right\rangle .
\end{aligned} \tag{10}
$$

In the limit shown, where $N$, the total number of replicas, tends to zero, equation (10) becomes equal to $C(\infty)$ given by equation (8). This completes the proof of the equivalence of the long-time and replica correlations.

We note, in conclusion, that the construction of replicas gives rise, in general, to an auxiliary field, analogous to the internal or molecular field of Curie–Weiss theory. This analogy between selfconsistent field methods and the use of replication will be discussed at greater length in a separate paper, with reference to models of magnets and liquid crystals.

## References

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